Solution of the Probabilistic Lambert Problem

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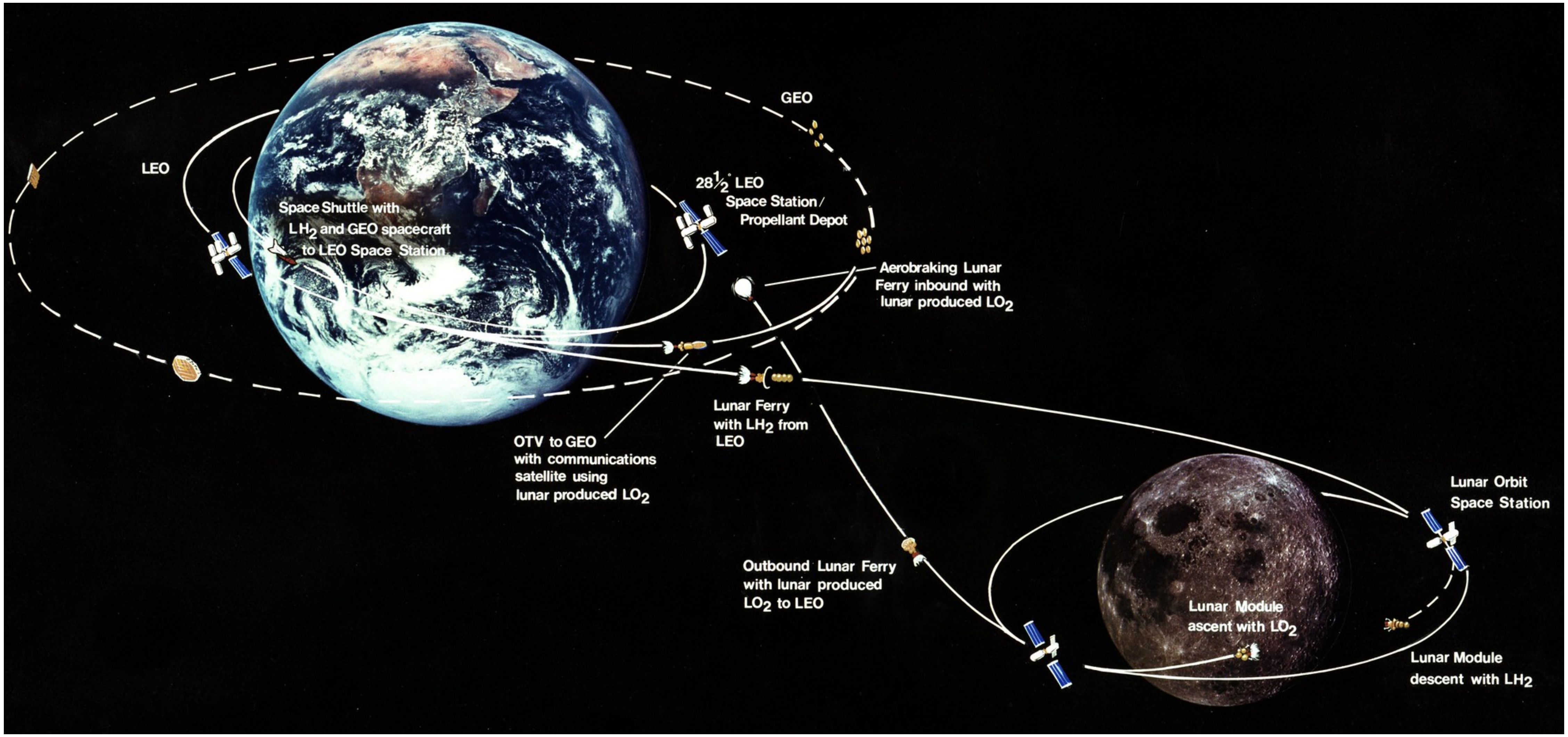
Lambert's Problem

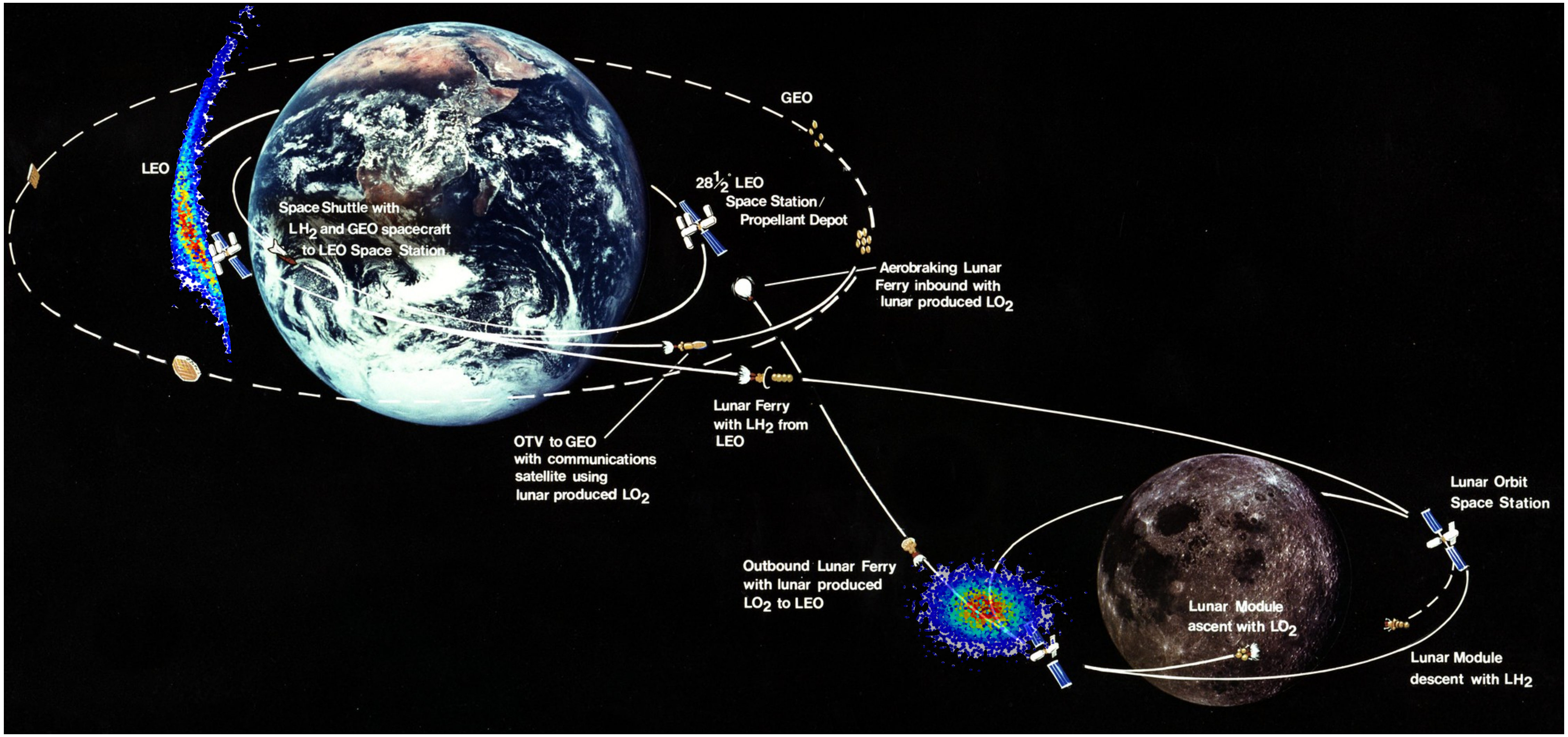


3D position coordinate $\boldsymbol{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Find velocity control policy $\dot{\boldsymbol{r}} := \boldsymbol{v}(t, \boldsymbol{r})$ such that

$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \quad \boxed{\boldsymbol{r}(t = t_0) = \boldsymbol{r}_0(\text{ given }), \quad \boldsymbol{r}(t = t_1) = \boldsymbol{r}_1(\text{ given })}$$

Probabilistic Lambert's Problem



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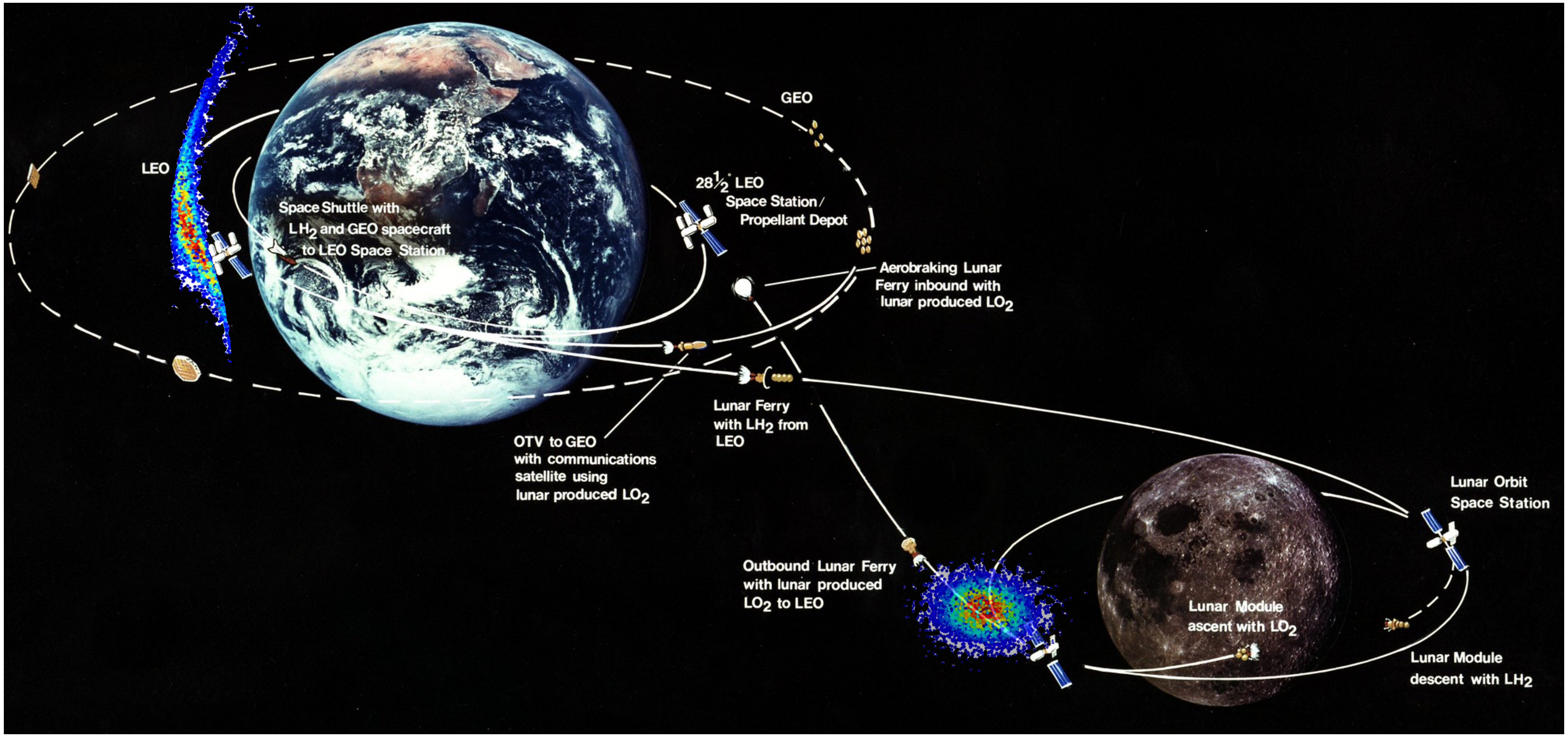

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$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \quad \boxed{\boldsymbol{r}(t = t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{r}(t = t_1) \sim \rho_1 \text{ (given)}}$$

Optimal Mass Transport (OMT)

Static formulation by Gaspard Monge in 1781:

$$\underset{\text{measurable } \boldsymbol{\tau}:\mathbb{R}^d \mapsto \mathbb{R}^d}{\arg\inf} \quad \mathbb{E}_{\rho_0} \|\boldsymbol{x} - \boldsymbol{y}\|_2^2$$

$$\text{subject to} \quad \boldsymbol{x} \sim \rho_0, \quad \boldsymbol{y} \sim \rho_1, \quad \boldsymbol{\tau}(\boldsymbol{x}) = \boldsymbol{y}$$

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Static reformulation by Kantorovich-Rubinstein in 1941:

$$\underset{\pi \in \Pi(\rho_0, \rho_1)}{\arg\inf} \quad \mathbb{E}_{\pi} \|\boldsymbol{x} - \boldsymbol{y}\|_2^2$$

$$\text{subject to} \quad \boldsymbol{x} \sim \rho_0, \quad \boldsymbol{y} \sim \rho_1$$

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Static reformulation by Kantorovich-Rubinstein in 1941:

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$$\text{subject to} \quad \boldsymbol{x} \sim \rho_0, \quad \boldsymbol{y} \sim \rho_1$$

Dynamic reformulation by Benamou-Brenier in 1999:

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \quad \int_{t_0}^{t_1} \mathbb{E}_{\rho}\left[\frac{1}{2}\|\boldsymbol{v}\|_2^2\right] \mathrm{d}t$$

$$\begin{aligned} \text{subject to} \quad & \dot{\boldsymbol{x}} = \boldsymbol{v}, \\ & \boldsymbol{x}(t=t_0) \sim \rho_0, \quad \boldsymbol{x}(t=t_1) \sim \rho_1 \end{aligned}$$

Connection with Optimal Control Problem (OCP)

Lambert Problem ⇔ Deterministic OCP

Reformulate Lambert's problem as deterministic OCP
[Bando and Yamakawa, AIAA JGCD, 2010]

$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \quad \boldsymbol{r}(t = t_0) = \boldsymbol{r}_0(\text{ given }), \quad \boldsymbol{r}(t = t_1) = \boldsymbol{r}_1(\text{ given })$$

$$\Updownarrow$$

$$\underset{\boldsymbol{v}}{\arg\inf} \int_{t_0}^{t_1} \left(\frac{1}{2}\|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r})\right)\mathrm{d}t$$

$$\dot{\boldsymbol{r}} = \boldsymbol{v},$$

$$\boldsymbol{r}(t = t_0) = \boldsymbol{r}_0 \text{ (given)}, \quad \boldsymbol{r}(t = t_1) = \boldsymbol{r}_1 \text{ (given)}$$

Our Contributions

Probabilistic Lambert Problem ⇔ Generalized OMT

$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \quad \boldsymbol{r}(t=t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{r}(t=t_1) \sim \rho_1 \text{ (given)}$$

$$\Updownarrow$$

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \quad \int_{t_0}^{t_1} \mathbb{E}_\rho \left[\frac{1}{2} \|\boldsymbol{v}\|_2^2 - \underbrace{V(\boldsymbol{r})}_{\text{Potential as state cost } (V=0 \text{ is OMT})} \right] \mathrm{d}t$$

$$\dot{\boldsymbol{r}} = \boldsymbol{v},$$

$$\boldsymbol{r}(t=t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{r}(t=t_1) \sim \rho_1 \text{ (given)}$$

Optimal Density Steering Problem

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \quad \int_{t_0}^{t_1} \mathbb{E}_{\rho}\left[\frac{1}{2}\|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r})\right] \mathrm{d}t$$

$$\dot{\boldsymbol{r}} = \boldsymbol{v},$$

$$\boldsymbol{r}(t=t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{r}(t=t_1) \sim \rho_1 \text{ (given)}$$

$$\Updownarrow$$

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \quad \int_{t_0}^{t_1} \mathbb{E}_{\rho}\left[\frac{1}{2}\|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r})\right] \mathrm{d}t$$

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) = 0,$$ — **Liouville PDE**

$$\rho(t=t_0,\cdot) = \rho_0, \quad \rho(t=t_1,\cdot) = \rho_1$$

Dynamic Stochastic Regularization

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \quad \int_{t_0}^{t_1} \mathbb{E}_\rho\left[\frac{1}{2}\|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r})\right] \mathrm{d}t$$

(Problem 1)

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{r}} \cdot (\rho\boldsymbol{v}) = 0,$$

Liouville PDE

$$\rho(t = t_0, \cdot) = \rho_0, \quad \rho(t = t_1, \cdot) = \rho_1$$

Generalized stochastic OMT a.k.a. generalized Schrödinger bridge problem

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \quad \int_{t_0}^{t_1} \mathbb{E}_\rho\left[\frac{1}{2}\|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r})\right] \mathrm{d}t$$

(Problem 2)

Small regularization > 0

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{r}} \cdot (\rho\boldsymbol{v}) = \varepsilon\Delta_{\boldsymbol{r}}\rho,$$

Fokker-Planck-Kolmogorov PDE

$$\rho(t = t_0, \cdot) = \rho_0, \quad \rho(t = t_1, \cdot) = \rho_1$$

Solution: Properties

Thm. (Solution consistency)

$$\underbrace{\left(\rho_{\varepsilon}^{\text{opt}}, \frac{v_{\varepsilon}^{\text{opt}}}{\sqrt{2\varepsilon}}\right)}_{\text{Solution of Problem 2}} \xrightarrow{\varepsilon\downarrow 0} \underbrace{\left(\rho^{\text{opt}}, v^{\text{opt}}\right)}_{\text{Solution of Problem 1}}$$

Thm. (Necessary conditions of optimality for Problem 2)

Value function

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\|\nabla_{\boldsymbol{r}}\psi\|_2^2 + \Delta_{\boldsymbol{r}}\psi = V(\boldsymbol{r}) \quad \text{— \textbf{HJB PDE}}$$

$$\frac{\partial \rho_{\varepsilon}^{\text{opt}}}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \left(\rho_{\varepsilon}^{\text{opt}} \nabla_{\boldsymbol{r}}\psi\right) = \varepsilon \Delta_{\boldsymbol{r}} \rho_{\varepsilon}^{\text{opt}}$$

$$\boldsymbol{v}_{\varepsilon}^{\text{opt}} = \nabla_{\boldsymbol{r}}\psi \quad \text{— \textbf{Optimal control}}$$

$$\rho_{\varepsilon}^{\text{opt}}(t = t_0, \cdot) = \rho_0, \quad \rho_{\varepsilon}^{\text{opt}}(t = t_1, \cdot) = \rho_1$$

Solution: Computation

Thm. (Hopf-Cole a.k.a. Fleming's log transform)

Change of variable $\left(\rho_{\varepsilon}^{\text{opt}}, \psi\right) \mapsto (\widehat{\varphi}, \varphi)$ — **Schrödinger factors**

$$\widehat{\varphi}(t, \boldsymbol{r}) = \rho_{\varepsilon}^{\text{opt}}(t, \boldsymbol{r}) \exp\left(-\frac{\psi(t, \boldsymbol{r})}{2\varepsilon}\right)$$

$$\varphi(t, \boldsymbol{r}) = \exp\left(\frac{\psi(t, \boldsymbol{r})}{2\varepsilon}\right)$$

results in a boundary-coupled system of forward-backward reaction-diffusion PDEs

$$\boxed{\begin{aligned} \frac{\partial \widehat{\varphi}}{\partial t} &= (\varepsilon \Delta_{\boldsymbol{r}} + V(\boldsymbol{r}))\widehat{\varphi} \longleftarrow \mathcal{L}_{\text{forward}}\widehat{\varphi} \\ \frac{\partial \varphi}{\partial t} &= -(\varepsilon \Delta_{\boldsymbol{r}} + V(\boldsymbol{r}))\varphi \longleftarrow \mathcal{L}_{\text{backward}}\varphi \\ \widehat{\varphi}(t = t_0, \cdot)\varphi(t = t_0, \cdot) = \rho_0, &\quad \widehat{\varphi}(t = t_1, \cdot)\varphi(t = t_1, \cdot) = \rho_1 \end{aligned}}$$

Optimally controlled joint state PDF: $\rho_{\varepsilon}^{\text{opt}}(t, \boldsymbol{r}) = \widehat{\varphi}(t, \boldsymbol{r})\varphi(t, \boldsymbol{r})$

Optimal control: $\boldsymbol{v}_{\varepsilon}^{\text{opt}}(t, \boldsymbol{r}) = 2\varepsilon \nabla_{\boldsymbol{r}} \log \varphi(t, \boldsymbol{r})$

Solution: Computation (contd.)

IDEA: Fixed point recursion over pair $\left(\varphi_1, \hat{\varphi}_0\right)$

$$\begin{tikzcd}
\hat{\varphi}_0(\boldsymbol{x}) \arrow[r, "\int \mathcal{L}_{\text{forward}}"] & \hat{\varphi}_1(\boldsymbol{x}) \arrow[d, "\rho_1(\boldsymbol{x}) \oslash \hat{\varphi}_1(\boldsymbol{x})"] \\
\varphi_0(\boldsymbol{x}) \arrow[u, "\rho_0(\boldsymbol{x}) \oslash \varphi_0(\boldsymbol{x})"] & \varphi_1(\boldsymbol{x}) \arrow[l, "\int \mathcal{L}_{\text{backward}}"]
\end{tikzcd}$$

Thm. (Existence-uniqueness-convergence) Proof by contraction mapping

Thm. (Fredholm Integral Representation)

$$\widehat{\varphi}(t,\boldsymbol{x}) = \underbrace{\frac{1}{\sqrt{(4\pi\varepsilon t)^3}} \int_{\mathbb{R}^3} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|_2^2}{4\varepsilon t}\right) \widehat{\varphi}_0(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}}_{\text{term 1}}$$

$$+ \underbrace{\int_0^t \frac{1}{\sqrt{(4\pi\varepsilon(t-\tau))^3}} \int_{\mathbb{R}^3} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|_2^2}{4\varepsilon(t-\tau)}\right) V(\boldsymbol{y})\,\widehat{\varphi}(\tau,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\tau}_{\text{term 2}}$$

Likewise for $\varphi(t,\boldsymbol{x})$

Numerical Case Study

Using left (for forward) or right (for backward) endpoint integral approximation for the Schrödinger factor IVP solutions

Prescribed time horizon $\left[t_0, t_1\right] \equiv \left[0,1\right]$

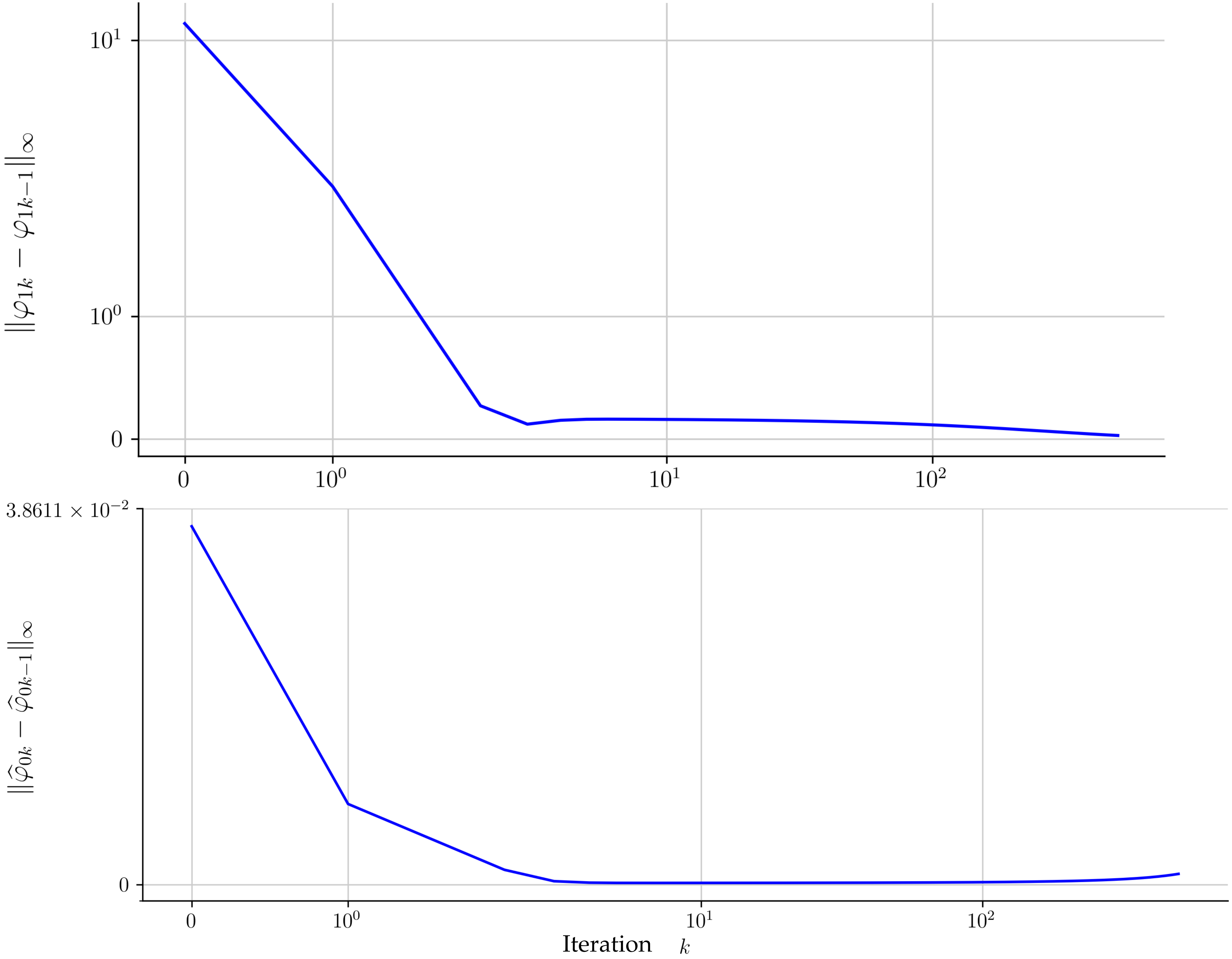
Potential $V(\boldsymbol{r}) = -\dfrac{\mu}{\|\boldsymbol{r}\|_2} = -\dfrac{\mu}{\sqrt{x^2+y^2+z^2}}$, gravitational constant $\mu > 0$

Endpoint joint PDFs

$$\begin{aligned}\rho_0 &= 0.5\mathcal{N}(\mathbf{0}, \boldsymbol{I}) + 0.5\mathcal{N}(\mathbf{1}, \boldsymbol{I})\\ \rho_1 &= \mathcal{N}(\mathbf{0}, \boldsymbol{I})\end{aligned}$$

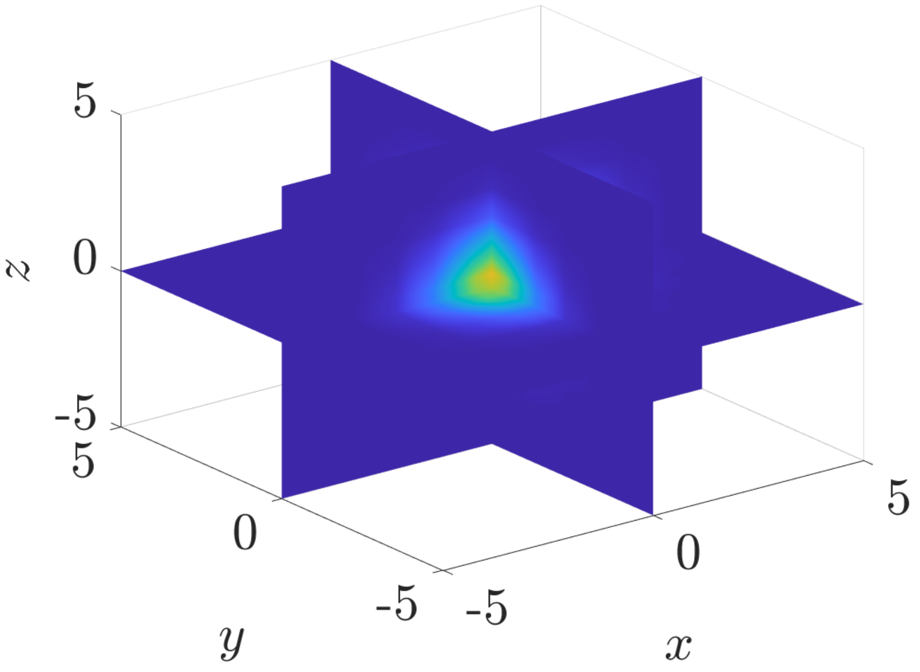
Numerical Case Study (contd.)

Recursions over pair $\left(\varphi_1, \hat{\varphi}_0\right)$

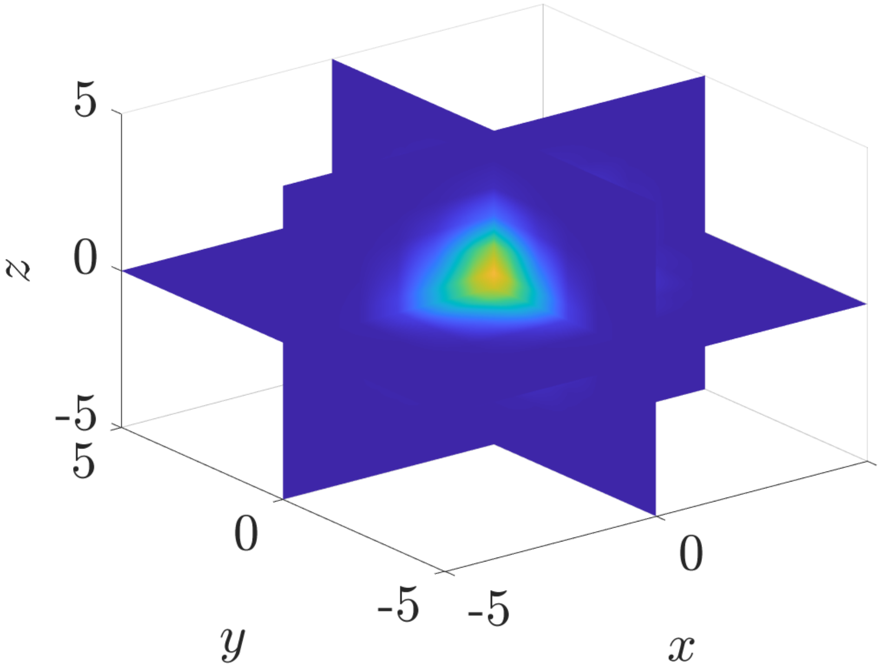


Numerical Case Study (contd.)

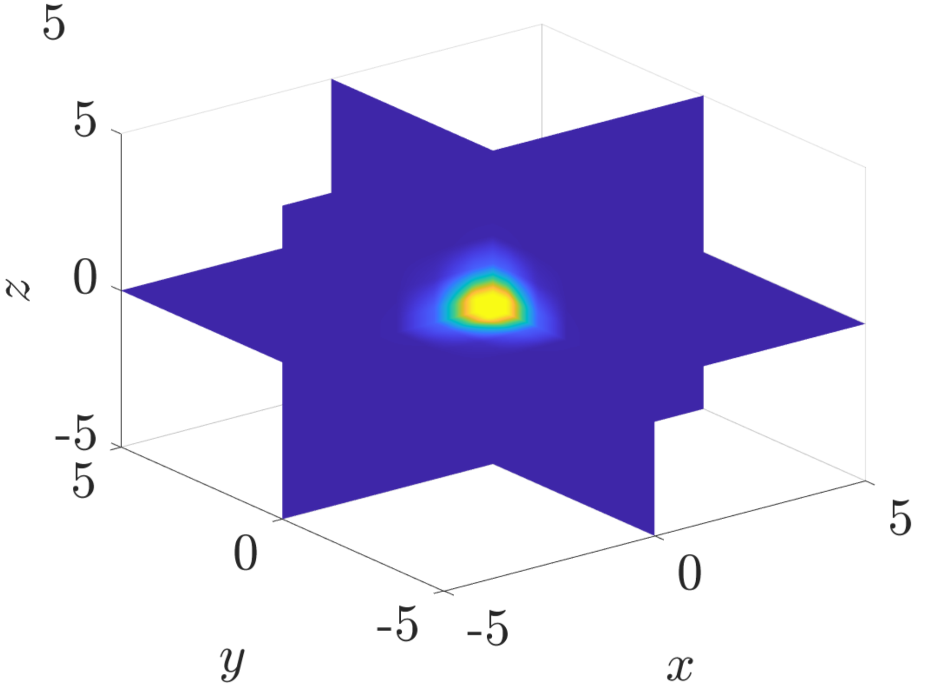
Slices of optimally controlled joint PDFs



$t = 0.01$



$t = 0.49$



$t = 1.00$

Thank You